

SOME HERMITE-HADAMARD TYPE INEQUALITIES FOR THE PRODUCT OF TWO OPERATOR PREINVEX FUNCTIONS

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ABSTRACT. In this paper we introduce operator preinvex functions and establish a Hermite-Hadamard type inequality for such functions. We give an estimate of the right hand side of a Hermite-Hadamard type inequality in which some operator preinvex functions of selfadjoint operators in Hilbert spaces are involved. Also some Hermite-Hadamard type inequalities for the product of two operator preinvex functions are given.

1. INTRODUCTION

The following inequality holds for any convex function f defined on \mathbb{R} and $a, b \in \mathbb{R}$, with $a < b$

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2} \quad (1.1)$$

Both inequalities hold in the reversed direction if f is concave. We note that Hermite-Hadamard's inequality may be regarded as a refinement of the concept of convexity and it follows easily from Jensen's inequality. The classical Hermite-Hadamard inequality provides estimates of the mean value of a continuous convex function $f : [a, b] \rightarrow \mathbb{R}$. The Hermite-Hadamard inequality has several applications in nonlinear analysis and the geometry of Banach spaces, see [12, 6].

In recent years several extensions and generalizations have been considered for classical convexity. We would like to refer the reader to [4, 8, 17] and references therein for more information. A number of papers have been written on this inequality providing some inequalities analogous to Hadamard's inequality given in (1.1) involving two convex functions, see [15, 2, 16]. Pachpatte in [15] has proved the following theorem for the product of two convex functions.

Theorem 1.1. *Let f and g be real-valued, nonnegative and convex functions on $[a, b]$. Then*

$$\frac{1}{b-a} \int_a^b f(x)g(x)dx \leq \frac{1}{3}M(a, b) + \frac{1}{6}N(a, b),$$

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$$2f\left(\frac{a+b}{2}\right)g\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(x)g(x)dx + \frac{1}{6}M(a,b) + \frac{1}{3}N(a,b),$$

where $M(a,b) = f(a)g(a) + f(b)g(b)$, $N(a,b) = f(a)g(b) + f(b)g(a)$.

A significant generalization of convex functions is that of invex functions introduced by Hanson in [11]. In this paper we introduce operator preinvex functions and give an operator version of the Hermite-Hadamard inequality for such functions.

First, we review the operator order in $B(H)$ and the continuous functional calculus for a bounded selfadjoint operator. For selfadjoint operators $A, B \in B(H)$ we write $A \leq B$ (or $B \geq A$) if $\langle Ax, x \rangle \leq \langle Bx, x \rangle$ for every vector $x \in H$, we call it the operator order.

Now, let A be a bounded selfadjoint linear operator on a complex Hilbert space $(H; \langle \cdot, \cdot \rangle)$ and $C(Sp(A))$ the C^* -algebra of all continuous complex-valued functions on the spectrum of A . The Gelfand map establishes a $*$ -isometrically isomorphism Φ between $C(Sp(A))$ and the C^* -algebra $C^*(A)$ generated by A and the identity operator 1_H on H as follows (see for instance [10, p.3]): For $f, g \in C(Sp(A))$ and $\alpha, \beta \in \mathbb{C}$

$$(i) \quad \Phi(\alpha f + \beta g) = \alpha \Phi(f) + \beta \Phi(g);$$

$$(ii) \quad \Phi(fg) = \Phi(f)\Phi(g) \text{ and } \Phi(f^*) = \Phi(f)^*;$$

$$(iii) \quad \|\Phi(f)\| = \|f\| := \sup_{t \in Sp(A)} |f(t)|;$$

$$(iv) \quad \Phi(f_0) = 1 \text{ and } \Phi(f_1) = A, \text{ where } f_0(t) = 1 \text{ and } f_1(t) = t, \text{ for } t \in Sp(A).$$

If f is a continuous complex-valued functions on $Sp(A)$, the element $\Phi(f)$ of $C^*(A)$ is denoted by $f(A)$, and we call it the continuous functional calculus for a bounded selfadjoint operator A .

If A is a bounded selfadjoint operator and f is a real-valued continuous function on $Sp(A)$, then $f(t) \geq 0$ for any $t \in Sp(A)$ implies that $f(A) \geq 0$, i.e., $f(A)$ is a positive operator on H . Moreover, if both f and g are real-valued functions on $Sp(A)$ such that $f(t) \leq g(t)$ for any $t \in sp(A)$, then $f(A) \leq g(A)$ in the operator order in $B(H)$.

A real valued continuous function f on an interval I is said to be operator convex (operator concave) if

$$f((1-\lambda)A + \lambda B) \leq (\geq) (1-\lambda)f(A) + \lambda f(B)$$

in the operator order in $B(H)$, for all $\lambda \in [0, 1]$ and for every bounded self-adjoint operators A and B in $B(H)$ whose spectra are contained in I .

For some fundamental results on operator convex (operator concave) and operator monotone functions, see [10, 5] and the references therein.

Dragomir in [7] has proved a Hermite-Hadamard type inequality for operator convex functions:

Theorem 1.2. *Let $f : I \rightarrow \mathbb{R}$ be an operator convex function on the interval I . Then for any selfadjoint operators A and B with spectra in I we have the*

inequality

$$\begin{aligned}
\left(f\left(\frac{A+B}{2}\right) \leq \right) & \frac{1}{2} \left[f\left(\frac{3A+B}{4}\right) + f\left(\frac{A+3B}{4}\right) \right] \\
& \leq \int_0^1 f((1-t)A + tB) dt \\
& \leq \frac{1}{2} \left[f\left(\frac{A+B}{2}\right) + \frac{f(A) + f(B)}{2} \right] \left(\leq \frac{f(A) + f(B)}{2} \right).
\end{aligned}$$

Moslehian in [14] generalized the above theorem 1.2 as follows:

Theorem 1.3. *If A, B are self-adjoint operators on a Hilbert space H with spectra in an interval J , f is an operator convex function on J and k, p are positive integers, then*

$$\begin{aligned}
f\left(\frac{A+B}{2}\right) & \leq \frac{1}{k^p} \sum_{i=0}^{k^p-1} f\left(\frac{2i+1}{2k^p}A + \left(1 - \frac{2i+1}{2k^p}\right)B\right) \\
& \leq \int_0^1 f((1-t)A + tB) dt \\
& \leq \frac{1}{2k^p} \sum_{i=0}^{k^p-1} \left[f\left(\frac{i+1}{k^p}A + \left(1 - \frac{i+1}{k^p}\right)B\right) + f\left(\frac{i}{k^p}A + \left(1 - \frac{i}{k^p}\right)B\right) \right] \\
& \leq \frac{f(A) + f(B)}{2}.
\end{aligned}$$

Motivated by the above results we investigate in this paper the operator version of the Hermite-Hadamard inequality for operator preinvex functions. We show that Theorem 1.3 holds for operator preinvex functions and establish an estimate of the right hand side of a Hermite-Hadamard type inequality in which some operator preinvex functions of selfadjoint operators in Hilbert spaces are involved. We also give some Hermite-Hadamard type inequalities for the product of two operator preinvex functions.

2. OPERATOR PREINVEX FUNCTIONS

Definition 2.1. Let X be a real vector space, a set $S \subseteq X$ is said to be invex with respect to the map $\eta : S \times S \rightarrow X$, if for every $x, y \in S$ and $t \in [0, 1]$,

$$y + t\eta(x, y) \in S. \quad (2.1)$$

It is obvious that every convex set is invex with respect to the map $\eta(x, y) = x - y$, but there exist invex sets which are not convex (see [1]).

Let $S \subseteq X$ be an invex set with respect to $\eta : S \times S \rightarrow X$. For every $x, y \in S$ the η -path P_{xv} joining the points x and $v := x + \eta(y, x)$ is defined as follows

$$P_{xv} := \{z : z = x + t\eta(y, x) : t \in [0, 1]\}.$$

The mapping η is said to be satisfies the condition C if for every $x, y \in S$ and $t \in [0, 1]$,

$$(C) \quad \begin{aligned} \eta(y, y + t\eta(x, y)) &= -t\eta(x, y), \\ \eta(x, y + t\eta(x, y)) &= (1 - t)\eta(x, y). \end{aligned}$$

Note that for every $x, y \in S$ and every $t_1, t_2 \in [0, 1]$ from condition C we have

$$\eta(y + t_2\eta(x, y), y + t_1\eta(x, y)) = (t_2 - t_1)\eta(x, y), \quad (2.2)$$

see [13, 18] for details.

Let \mathcal{A} be a C^* -algebra, denote by \mathcal{A}_{sa} the set of all self adjoint elements in \mathcal{A} .

Definition 2.2. Let I be an interval in \mathbb{R} and $S \subseteq B(H)_{sa}$ be an invex set with respect to $\eta : S \times S \rightarrow B(H)_{sa}$. A continuous function $f : I \rightarrow \mathbb{R}$ is said to be operator preinvex on I with respect to η for operators in S if

$$f(B + t\eta(A, B)) \leq (1 - t)f(B) + tf(A). \quad (2.3)$$

in the operator order in $B(H)$, for all $t \in [0, 1]$ and for every $A, B \in S$ whose spectra are contained in I .

Every operator convex function is an operator preinvex with respect to the map $\eta(A, B) = A - B$ but the converse does not holds (see the following example).

Now, we give an example of some operator preinvex functions and invex sets with respect to the maps η which satisfy the conditions (C).

Example 2.3. (a) Suppose that 1_H is the identity operator on a Hilbert space H , and

$$\begin{aligned} T &:= \{A \in B(H)_{sa} : A \leq -1 \times 1_H\} \\ U &:= \{A \in B(H)_{sa} : 1_H \leq A\} \\ S &:= T \cup U \subseteq B(H)_{sa}. \end{aligned}$$

Suppose that the function $\eta_1 : S \times S \rightarrow B(H)_{sa}$ is defined by

$$\eta_1(A, B) = \begin{cases} A - B & A, B \in U, \\ A - B & A, B \in T, \\ 1_H - B & A \in T, B \in U, \\ -1_H - B & A \in U, B \in T. \end{cases}$$

Clearly η_1 satisfies condition C and S is an invex set with respect to η_1 . We show that the real function $f(t) = t^2$ is operator preinvex with respect to η_1 on every interval $I \subseteq \mathbb{R}$, for operators in S . Since f is an operator convex function on I , for the cases which $\eta_1(A, B) = A - B$ the inequality (2.3) holds. Let $\eta_1(A, B) = 1_H - B$, in this case we have $1_H \leq -A \leq A^2$ and

$$\begin{aligned} (B + t\eta_1(A, B))^2 &= (B + t(1_H - B))^2 = ((1 - t)B + t1_H)^2 \\ &\leq (1 - t)B^2 + t1_H \leq (1 - t)B^2 + tA^2. \end{aligned}$$

Similarly, for the case $\eta_1 = -1_H - B$ we have

$$\begin{aligned} (B + t\eta_1(A, B))^2 &= (B + t(-1_H - B))^2 = ((1-t)(-B) + t1_H)^2 \\ &\leq (1-t)B^2 + t1_H \leq (1-t)B^2 + tA^2, \end{aligned}$$

therefore, the inequality (2.3) holds.

But the real function $g(t) = a + bt$, $a, b \in \mathbb{R}$ is not operator preinvex with respect to η_1 on S .

- (b) Suppose that $V := (-2 \times 1_H, 0)$, $W := (0, 2 \times 1_H)$, $S := V \cup W \subseteq B(H)_{sa}$ and the function $\eta_2 : S \times S \rightarrow B(H)_{sa}$ is defined by

$$\eta_2(A, B) = \begin{cases} A - B & A, B \in V \text{ or } A, B \in W, \\ 0 & \text{otherwise.} \end{cases}$$

Clearly η_2 satisfies condition C and S is an invex set with respect to η_2 . The constant functions $f(t) = a$, $a \in \mathbb{R}$ is only operator preinvex functions with respect to η_2 for operators in S . Because for $\eta_2 = 0$,

$$f(B + t\eta_2(A, B)) = f(B) \leq (1-t)f(B) + tf(A),$$

implies that $f(A) - f(B) \geq 0$. interchanging A, B we get $f(B) - f(A) \geq 0$.

- (c) The function $f(t) = -|t|$ is not a convex function, but it is a operator preinvex function with respect to η_3 , where

$$\eta_3(A, B) = \begin{cases} A - B & A, B \geq 0 \text{ or } A, B \leq 0, \\ B - A & \text{otherwise } A \leq 0 \leq B \text{ or } B \leq 0 \leq A. \end{cases}$$

Let X be a vector space, $x, y \in X$, $x \neq y$. Define the segment

$$[x, y] := (1-t)x + ty; t \in [0, 1].$$

We consider the function $f : [x, y] \rightarrow \mathbb{R}$ and the associated function

$$\begin{aligned} g(x, y) &: [0, 1] \rightarrow \mathbb{R}, \\ g(x, y)(t) &:= f((1-t)x + ty), t \in [0, 1]. \end{aligned}$$

Note that f is convex on $[x, y]$ if and only if $g(x, y)$ is convex on $[0, 1]$. For any convex function defined on a segment $[x, y] \in X$, we have the Hermite- Hadamard integral inequality

$$f\left(\frac{x+y}{2}\right) \leq \int_0^1 f((1-t)x + ty)dt \leq \frac{f(x) + f(y)}{2}, \quad (2.4)$$

which can be derived from the classical Hermite-Hadamard inequality (1.1) for the convex function $g(x, y) : [0, 1] \rightarrow \mathbb{R}$.

Proposition 2.4. *Let I be an interval in \mathbb{R} , $S \subseteq B(H)_{sa}$ an invex set with respect to $\eta : S \times S \rightarrow B(H)_{sa}$ and η satisfies condition C on S and $f : I \rightarrow \mathbb{R}$ a continuous function. Then, for every $A, B \in S$ with spectra of $A, V := A + \eta(B, A)$ in I , the function f is operator preinvex on I with respect to η for operators in η -path P_{AV} if and only if the function $\varphi_{x,A,B} : [0, 1] \rightarrow \mathbb{R}$ defined by*

$$\varphi_{x,A,B}(t) := \langle f(A + t\eta(B, A))x, x \rangle \quad (2.5)$$

is convex on $[0, 1]$ for every $x \in H$ with $\|x\| = 1$.

Proof. Let the function f be operator preinvex on I with respect to η for operators in η -path P_{AV} . Suppose that $A, B \in S$ with spectra A, V in I , since $A + t\eta(B, A) = tV + (1 - t)A$ therefore for all $t \in [0, 1]$, $Sp(A + t\eta(B, A)) \subseteq I$. If $t_1, t_2 \in [0, 1]$ since η satisfies condition C on S , we have

$$\begin{aligned} \varphi_{x,A,B}((1 - \lambda)t_1 + \lambda t_2) &= \langle f(A + ((1 - \lambda)t_1 + \lambda t_2)\eta(B, A))x, x \rangle \\ &= \langle f(A + t_1\eta(B, A) + \lambda\eta(A + t_2\eta(B, A), A + t_1\eta(B, A)))x, x \rangle \\ &\leq \lambda \langle f(A + t_2\eta(B, A))x, x \rangle + (1 - \lambda) \langle f(A + t_1\eta(B, A))x, x \rangle \\ &= \lambda \varphi_{x,A,B}(t_2) + (1 - \lambda) \varphi_{x,A,B}(t_1), \end{aligned} \tag{2.6}$$

for every $\lambda \in [0, 1]$ and $x \in H$ with $\|x\| = 1$. Therefore, $\varphi_{x,A,B}$ is convex on $[0, 1]$.

Conversely, suppose that $x \in H$ with $\|x\| = 1$ and $\varphi_{x,A,B}$ is convex on $[0, 1]$ and $C_1 := A + t_1\eta(B, A) \in P_{AV}$, $C_2 := A + t_2\eta(B, A) \in P_{AV}$. Fix $\lambda \in [0, 1]$. By (2.4) we have

$$\begin{aligned} \langle f(C_1 + \lambda\eta(C_2, C_1))x, x \rangle &= \langle f(A + ((1 - \lambda)t_1 + \lambda t_2)\eta(B, A))x, x \rangle \\ &= \varphi_{x,A,B}((1 - \lambda)t_1 + \lambda t_2) \\ &\leq (1 - \lambda) \varphi_{x,A,B}(t_1) + \lambda \varphi_{x,A,B}(t_2) \\ &= (1 - \lambda) \langle f(C_1)x, x \rangle + \lambda \langle f(C_2)x, x \rangle. \end{aligned} \tag{2.7}$$

Hence, f is operator preinvex with respect to η for operators in η -path P_{AV} . \square

3. HERMITE-HADAMARD TYPE INEQUALITIES

In this section we generalize Theorem 1.1 and Theorem 1.3 for operator preinvex functions and establish an estimate for the right-hand side of the Hermite-Hadamard operator inequality for such functions. Some Hermite-Hadamard type inequalities for the product of two operator preinvex functions is also given.

The following Theorem is a generalization of Theorem 1.3 for operator preinvex functions.

Theorem 3.1. *Let $S \subseteq B(H)_{sa}$ be an invex set with respect to $\eta : S \times S \rightarrow B(H)_{sa}$ and η satisfies condition C . If for every $A, B \in S$ with spectra of $A, V := A + \eta(B, A)$ in the interval I , the function $f : I \rightarrow \mathbb{R}$ is operator preinvex with respect to η for operators in η -path P_{AV} and k, p are positive integers, then the*

following inequalities holds

$$\begin{aligned}
 f\left(A + \frac{1}{2}\eta(B, A)\right) &\leq \frac{1}{k^p} \sum_{i=0}^{k^p-1} f\left(A + \frac{2i+1}{2k^p}\eta(B, A)\right) \\
 &\leq \int_0^1 f(A + t\eta(B, A))dt \\
 &\leq \frac{1}{2k^p} \sum_{i=0}^{k^p-1} \left[f\left(A + \frac{i+1}{k^p}\eta(B, A)\right) + f\left(A + \frac{i}{k^p}\eta(B, A)\right) \right] \\
 &\leq \frac{f(A) + f(B)}{2}. \quad (3.1)
 \end{aligned}$$

Proof. For $x \in H$ with $\|x\| = 1$ and $t \in [0, 1]$, we have

$$\langle (A + t\eta(B, A))x, x \rangle = (1-t)\langle Ax, x \rangle + t\langle Vx, x \rangle \in I, \quad (3.2)$$

since $\langle Ax, x \rangle \in Sp(A) \subseteq I$ and $\langle Vx, x \rangle \in Sp(V) \subseteq I$.

Continuity of f and (3.2) imply that the operator valued integral $\int_0^1 f(A + t\eta(B, A))dt$ exists. Since η satisfied condition C , therefore by (2.2) for every $t \in [0, 1]$ we have

$$A + \frac{1}{2}\eta(B, A) = A + t\eta(B, A) + \frac{1}{2}\eta(A + (1-t)\eta(B, A), A + t\eta(B, A)). \quad (3.3)$$

Let $x \in H$ be a unit vector, define the real-valued function $\varphi : [0, 1] \rightarrow \mathbb{R}$ given by $\varphi(t) = \langle f(A + t\eta(B, A))x, x \rangle$. Since f is operator preinvex, by the previous proposition 2.4, φ is a convex function on $[0, 1]$. Utilizing the classical Hermite-Hadamard inequality for real-valued convex function φ on the interval $[\frac{i}{k^p}, \frac{i+1}{k^p}]$, we get

$$\varphi\left(\frac{2i+1}{2k^p}\right) \leq k^p \int_{\frac{i}{k^p}}^{\frac{i+1}{k^p}} \varphi(t)dt \leq \frac{\varphi(\frac{i}{k^p}) + \varphi(\frac{i+1}{k^p})}{2} \quad (3.4)$$

Summation of the above inequalities over $i = 0, \dots, k^p - 1$ yields

$$\sum_{i=0}^{k^p-1} \varphi\left(\frac{2i+1}{2k^p}\right) \leq k^p \int_0^1 \varphi(t)dt \leq \sum_{i=0}^{k^p-1} \frac{\varphi(\frac{i}{k^p}) + \varphi(\frac{i+1}{k^p})}{2}. \quad (3.5)$$

Hence

$$\begin{aligned}
 \frac{1}{k^p} \sum_{i=0}^{k^p-1} f\left(A + \frac{2i+1}{2k^p}\eta(B, A)\right) &\leq \int_0^1 f(A + t\eta(B, A))dt \\
 &\leq \frac{1}{2k^p} \sum_{i=0}^{k^p-1} \left[f\left(A + \frac{i+1}{k^p}\eta(B, A)\right) + f\left(A + \frac{i}{k^p}\eta(B, A)\right) \right]. \quad (3.6)
 \end{aligned}$$

Inequality (3.3) for $t = \frac{2i+1}{k^p}$ and operator preinvexity f imply that

$$\begin{aligned} & f\left(A + \frac{1}{2}\eta(B, A)\right) \\ & \leq \frac{1}{2}f\left(A + \frac{2i+1}{k^p}\eta(B, A)\right) + \frac{1}{2}f\left(A + \left(1 - \frac{2i+1}{k^p}\right)\eta(B, A)\right). \end{aligned} \quad (3.7)$$

Summation of the above inequalities over $i = 0, \dots, k^p - 1$ and the following equality

$$\sum_{i=0}^{k^p-1} f\left(A + \frac{2i+1}{k^p}\eta(B, A)\right) = \sum_{i=0}^{k^p-1} f\left(A + \left(1 - \frac{2i+1}{k^p}\right)\eta(B, A)\right)$$

yield

$$k^p f\left(A + \frac{1}{2}\eta(B, A)\right) \leq \sum_{i=0}^{k^p-1} f\left(A + \frac{2i+1}{k^p}\eta(B, A)\right). \quad (3.8)$$

In the other hand, from preinvexity f we have

$$\begin{aligned} & \frac{1}{2k^p} \sum_{i=0}^{k^p-1} \left[f\left(A + \frac{i+1}{k^p}\eta(B, A)\right) + f\left(A + \frac{i}{k^p}\eta(B, A)\right) \right] \\ & \leq \frac{1}{2k^p} \sum_{i=0}^{k^p-1} \left[\frac{i+1}{k^p} f(B) + \left(1 - \frac{i+1}{k^p}\right) f(A) + \frac{i}{k^p} f(B) + \left(1 - \frac{i}{k^p}\right) f(A) \right] \\ & \leq \frac{f(A) + f(B)}{2}. \end{aligned} \quad (3.9)$$

From inequalities (3.6), (3.8) and (3.9) we obtain (3.1). \square

A simple consequence of the above theorem is that the integral is closer to the left bound than to the right, namely we can state:

Corollary 3.2. *With the assumptions in Theorem 3.1 we have the inequality*

$$0 \leq \int_0^1 f(A + t\eta(B, A))dt - f\left(A + \frac{1}{2}\eta(B, A)\right) \leq \frac{f(A) + f(B)}{2} - \int_0^1 f(A + t\eta(B, A))dt.$$

Example 3.3. Let S , f , η_1 be as in Example 2.3, then we have

$$\left(A + \frac{1}{2}\eta_1(B, A)\right)^2 \leq \int_0^1 (A + t\eta_1(B, A))^2 dt \leq \frac{A^2 + B^2}{2},$$

for every $A, B \in S$.

Let $S \subseteq B(H)_{sa}$ be an invex set with respect to $\eta : S \times S \rightarrow B(H)_{sa}$ and $f, g : I \rightarrow \mathbb{R}$ operator preinvex functions on the interval I with respect to η for operators in η -path P_{AV} . Then for every $A, B \in S$ with spectra of $A, V := A + \eta(B, A)$ in the interval I , we define real functions $M(A, B)$ and $N(A, B)$ on H by

$$\begin{aligned} M(A, B)(x) &= \langle f(A)x, x \rangle \langle g(A)x, x \rangle + \langle f(B)x, x \rangle \langle g(B)x, x \rangle & (x \in H), \\ N(A, B)(x) &= \langle f(A)x, x \rangle \langle g(B)x, x \rangle + \langle f(B)x, x \rangle \langle g(A)x, x \rangle & (x \in H). \end{aligned}$$

The following Theorem is a generalization of Theorem 1.1 for operator preinvex functions.

Theorem 3.4. *Let $f, g : I \rightarrow \mathbb{R}^+$ be operator preinvex functions on the interval I with respect to η and η satisfies condition C. Then for any selfadjoint operators A and B on a Hilbert space H with spectra A, V in I , the inequality*

$$\begin{aligned} \int_0^1 \langle f(A + t\eta(B, A))x, x \rangle \langle g(A + t\eta(B, A))x, x \rangle dt \\ \leq \frac{1}{3}M(A, B)(x) + \frac{1}{6}N(A, B)(x), \end{aligned} \quad (3.10)$$

holds for any $x \in H$ with $\|x\| = 1$.

Proof. Continuity of f, g and (3.2) imply that the following operator valued integrals exist

$$\int_0^1 f(B + t\eta(B, A))dt, \int_0^1 g(A + t\eta(B, A))dt, \int_0^1 (fg)(A + t\eta(B, A))dt.$$

Since f and g are operator preinvex, therefore for t in $[0, 1]$ and $x \in H$ we have

$$\langle f(A + t\eta(B, A))x, x \rangle \leq \langle (tf(B) + (1-t)f(A))x, x \rangle, \quad (3.11)$$

$$\langle g(A + t\eta(B, A))x, x \rangle \leq \langle (tg(B) + (1-t)g(A))x, x \rangle. \quad (3.12)$$

From (3.11) and (3.12) we obtain

$$\begin{aligned} \langle f(A + t\eta(B, A))x, x \rangle \langle g(A + t\eta(B, A))x, x \rangle \\ \leq (1-t)^2 \langle f(A)x, x \rangle \langle g(A)x, x \rangle + t^2 \langle f(B)x, x \rangle \langle g(B)x, x \rangle \\ + t(1-t) [\langle f(A)x, x \rangle \langle g(B)x, x \rangle + \langle f(B)x, x \rangle \langle g(A)x, x \rangle]. \end{aligned} \quad (3.13)$$

Integrating both sides of (3.13) over $[0, 1]$ we get the required inequality (3.10). \square

Theorem 3.5. *Let $f, g : I \rightarrow \mathbb{R}$ be operator preinvex functions on the interval I with respect to η . If η satisfies condition C, then for any selfadjoint operators A and B on a Hilbert space H with spectra A, V in I , the inequality*

$$\begin{aligned} \left\langle f \left(A + \frac{1}{2}\eta(B, A) \right) x, x \right\rangle \left\langle g \left(A + \frac{1}{2}\eta(B, A) \right) x, x \right\rangle \\ \leq \frac{1}{2} \int_0^1 \langle f(A + t\eta(B, A))x, x \rangle \langle g(A + t\eta(B, A))x, x \rangle dt \\ + \frac{1}{12}M(A, B)(x) + \frac{1}{6}N(A, B)(x), \end{aligned} \quad (3.14)$$

holds for any $x \in H$ with $\|x\| = 1$.

Proof. Put $D = A + t\eta(B, A)$ and $E = A + (1 - t)\eta(B, A)$, by (3.3) we have $A + \frac{1}{2}\eta(B, A) = D + \frac{1}{2}\eta(E, D)$. Since f and g are operator preinvex, therefore for any $t \in I$ and any $x \in H$ with $\|x\| = 1$ we observe that

$$\begin{aligned}
& \left\langle f \left(A + \frac{1}{2}\eta(B, A) \right) x, x \right\rangle \left\langle g \left(A + \frac{1}{2}\eta(B, A) \right) x, x \right\rangle \\
&= \left\langle f \left(D + \frac{1}{2}\eta(E, D) \right) x, x \right\rangle \left\langle g \left(D + \frac{1}{2}\eta(E, D) \right) x, x \right\rangle \\
&\leq \left\langle \left(\frac{f(D) + f(E)}{2} \right) x, x \right\rangle \left\langle \left(\frac{g(D) + g(E)}{2} \right) x, x \right\rangle \\
&\leq \frac{1}{4} [\langle f(D)x, x \rangle + \langle f(E)x, x \rangle] [\langle g(D)x, x \rangle + \langle g(E)x, x \rangle] \\
&\leq \frac{1}{4} [\langle f(D)x, x \rangle \langle g(D)x, x \rangle + \langle f(E)x, x \rangle \langle g(E)x, x \rangle] \\
&+ \frac{1}{4} [t\langle f(A)x, x \rangle + (1-t)\langle f(B)x, x \rangle] [(1-t)\langle g(A)x, x \rangle + t\langle g(B)x, x \rangle] \\
&+ [(1-t)\langle f(A)x, x \rangle + t\langle f(B)x, x \rangle] [t\langle g(A)x, x \rangle + (1-t)\langle g(B)x, x \rangle] \\
&= \frac{1}{4} [\langle f(D)x, x \rangle \langle g(D)x, x \rangle + \langle f(E)x, x \rangle \langle g(E)x, x \rangle] \\
&+ \frac{1}{4} 2t(1-t) [\langle f(A)x, x \rangle \langle g(A)x, x \rangle + \langle f(B)x, x \rangle \langle g(B)x, x \rangle] \\
&+ (t^2 + (1-t)^2) [\langle f(A)x, x \rangle \langle g(B)x, x \rangle + \langle f(B)x, x \rangle \langle g(A)x, x \rangle]. \quad (3.15)
\end{aligned}$$

We integrate both sides of (3.15) over $[0,1]$ and obtain

$$\begin{aligned}
& \left\langle f \left(A + \frac{1}{2}\eta(B, A) \right) x, x \right\rangle \left\langle g \left(A + \frac{1}{2}\eta(B, A) \right) x, x \right\rangle \\
&\leq \frac{1}{4} \int_0^1 [\langle f(A + t\eta(B, A))x, x \rangle \langle g(A + t\eta(B, A))x, x \rangle \\
&+ \langle f(A + (1-t)\eta(B, A))x, x \rangle \langle g(A + (1-t)\eta(B, A))x, x \rangle] dt \\
&+ \frac{1}{12} M(A, B)(x) + \frac{1}{6} N(A, B)(x).
\end{aligned}$$

This implies the required inequality (3.14). \square

The following Theorem is a generalization of Theorem 3.1 in [3].

Theorem 3.6. *Let the function $f : I \rightarrow \mathbb{R}^+$ is continuous, $S \subseteq B(H)_{sa}$ be an open invex set with respect to $\eta : S \times S \rightarrow B(H)_{sa}$ and η satisfies condition C. If for every $A, B \in S$ and $V = A + \eta(B, A)$ the function f is operator preinvex with respect to η on η -path P_{AV} with spectra of A and V in I . Then, for every $a, b \in (0, 1)$ with $a < b$ and every $x \in H$ with $\|x\| = 1$ the following inequality*

holds,

$$\begin{aligned} & \left| \frac{1}{2} \left\langle \int_0^a f(A + s\eta(B, A)) ds \, x, x \right\rangle + \frac{1}{2} \left\langle \int_0^b f(A + s\eta(B, A)) ds \, x, x \right\rangle \right. \\ & \quad \left. - \frac{1}{b-a} \int_a^b \left\langle \int_0^t f(A + s\eta(B, A)) ds \, x, x \right\rangle dt \right| \\ & \leq \frac{b-a}{8} \{ \langle f(A + a\eta(B, A))x, x \rangle + \langle f(A + b\eta(B, A))x, x \rangle \}. \end{aligned} \quad (3.16)$$

Moreover we have

$$\begin{aligned} & \left\| \frac{1}{2} \int_0^a f(A + s\eta(B, A)) ds + \frac{1}{2} \int_0^b f(A + s\eta(B, A)) ds \right. \\ & \quad \left. - \frac{1}{b-a} \int_a^b \int_0^t f(A + s\eta(B, A)) ds dt \right\| \\ & \leq \frac{b-a}{8} \|f(A + a\eta(B, A)) + f(A + b\eta(B, A))\| \\ & \leq \frac{b-a}{8} [\|f(A + a\eta(B, A))\| + \|f(A + b\eta(B, A))\|]. \end{aligned} \quad (3.17)$$

Proof. Let $A, B \in S$ and $a, b \in (0, 1)$ with $a < b$. For $x \in H$ with $\|x\| = 1$ we define the function $\varphi : [0, 1] \rightarrow \mathbb{R}^+$ by

$$\varphi(t) := \left\langle \int_0^t f(A + s\eta(B, A)) ds \, x, x \right\rangle.$$

Utilizing the continuity of the function f , the continuity property of the inner product and the properties of the integral of operator-valued functions we have

$$\left\langle \int_0^t f(A + s\eta(B, A)) ds \, x, x \right\rangle = \int_0^t \langle f(A + s\eta(B, A)) \, x, x \rangle ds.$$

Since $f(A + s\eta(B, A)) \geq 0$, therefore $\varphi(t) \geq 0$ for all $t \in I$. Obviously for every $t \in (0, 1)$ we have

$$\varphi'(t) = \langle f(A + t\eta(B, A))x, x \rangle \geq 0,$$

hence, $|\varphi'(t)| = \varphi'(t)$. Since f is operator preinvex with respect to η on η -path P_{AV} , by Proposition 2.4 the function φ' is convex. Applying Theorem 2.2 in [9] to the function φ implies that

$$\left| \frac{\varphi(a) + \varphi(b)}{2} - \frac{1}{b-a} \int_a^b \varphi(s) ds \right| \leq \frac{(b-a)(\varphi'(a) + \varphi'(b))}{8},$$

and we deduce that (3.16) holds. Taking supremum over both side of inequality (3.16) for all x with $\|x\| = 1$, we deduce that the inequality (3.17) holds. \square

REFERENCES

1. T. Antczak, *Mean value in invexity analysis*, Nonlinear Analysis **60** (2005), 1471-1484.
2. M. Klarić Bakula and J. Pečarić *Note on some Hadamard-type inequalities*, J. Inequal. Pure Appl. Math. **5** (2004), no. 3, Article 74.

3. A. Barani, A.G. Ghazanfari, S.S. Dragomir, *Hermite-Hadamard inequality for functions whose derivatives absolute values are preinvex*, J. Inequal. Appl. (2012), Article ID 247.
4. N.S. Barnett, P. Cerone and S.S. Dragomir, *Some new inequalities for Hermite-Hadamard divergence in information theory*, Stochastic analysis and applications. Vol. 3, 7-19, Nova Sci. Publ., Hauppauge, NY, 2003.
5. R. Bhatia, *Matrix analysis*, Springer-Verlag, New York, 1997.
6. C. Conde, *A version of the Hermite-Hadamard inequality in a nonpositive curvature space*, Banach J. Math. Anal. **6** (2012), no. 2, 159-167.
7. S.S. Dragomir, *The Hermite-Hadamard type inequalities for operator convex functions*, Appl. Math. Comput. **218** (2011), no. 3, 766-772.
8. S.S. Dragomir and C.E.M. Pearce, *Selected Topics on Hermite-Hadamard Inequalities and applications*, (RGMIA Monographs [http:// rgmia.vu.edu.au/ monographs/ hermite hadamard.html](http://rgmia.vu.edu.au/monographs/hermitehadamard.html)), Victoria University, 2000.
9. S.S. Dragomir, and R.P. Agarwal, *Two inequalities for differentiable mappings and applications to special means of real numbers and to trapezoidal formula*, Appl. Math. Lett. **11** (1998), no. 5, 91-95.
10. T. Furuta, J. Mićić Hot, J. Pečarić and Y. Seo, *Mond-Pečarić Method in Operator Inequalities. Inequalities for Bounded Selfadjoint Operators on a Hilbert Space*, Element, Zagreb, 2005.
11. M.A. Hanson, *On sufficiency of the Kuhn-Tucker conditions*, J. Math. Anal. Appl. **80** (1981), 545-550.
12. E. Kikianty, *Hermite-Hadamard inequality in the geometry of Banach spaces*, PhD thesis, Victoria University, 2010.
13. S. R. Mohan and S. K. Neogy, *On invex sets and preinvex function*, J. Math. Anal. Appl. **189** (1995), 901-908.
14. M. S .Moslehian, *Matrix Hermite-Hadamard type inequalities*, Houston J. math **39** (2013) no 1, 178-189.
15. B. G. Pachpatte, *On some inequalities for convex functions*, RGMIA Res. Rep. Coll. (E) **6**, (2003).
16. M.Tunç, *On some new inequalities for convex functions*, Turk. J. Math. **36** (2012), 245-251.
17. S.Wu, *On the weighted generalization of the Hermite-Hadamard inequality and its applications*, Rocky Mountain J. Math. **39** (2009), no. 5, 1741-1749.
18. X. M. Yang and D. Li, *On properties of preinvex functions*, J. Math. Anal. Appl. **256** (2001), 229-241.

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